

The Critical Numbers of Rankin-Selberg Convolutions of Cohomological Representations

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Abstract

We study the critical numbers of the Rankin-Selberg convolution of arbitrary pairs of cohomological cuspidal automorphic representations and we parametrize these critical numbers by certain 1-dimensional subrepresentations attached to the corresponding pair of finite dimensional representations of the related general linear groups.

1 Introduction

For arbitrary natural numbers n and m let π and σ denote cuspidal automorphic representations of $GL_n(\mathbb{A})$ and $GL_m(\mathbb{A})$ respectively over the adèle ring \mathbb{A} of \mathbb{Q} . Jacquet, Piatetski-Shapiro and Shalika [4] introduced for such pairs (π, σ) an L -function $L(\pi, \sigma, s)$ which up to a few special cases is an entire function of the variable s . In analogy with Deligne's notion of *critical values* of motivic L -functions [2] we would like to study the values of $L(\pi, \sigma, s)$ at *critical numbers* $t \in \frac{n-m}{2} + \mathbb{Z}$ and in particular their arithmetic properties. Assuming that π and σ are cohomological in the case $m = n - 1$ these critical values are quite well understood by [5]. See also Januszewski's contributions [6] for totally real number fields. The assumption says that there are finite-dimensional irreducible rational representations M_μ and M_ν of GL_n and GL_m respectively of highest weights μ and ν with a certain purity property such that for the infinity components π_∞ and σ_∞ the representations $\pi_\infty \otimes M_\mu$ and $\sigma_\infty \otimes M_\nu$ have non-trivial relative Lie algebra cohomology, i.e. we have

$$H^\bullet(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}}) \neq 0$$

and

$$H^\bullet(\mathfrak{gl}_m, K_{m,\infty}; \sigma_\infty \otimes M_{\nu,\mathbb{C}}) \neq 0,$$

where for a natural number n as usual \mathfrak{gl}_n denotes the Lie algebra of $GL_n(\mathbb{R})$, $K_{n,\infty} = SO_n(\mathbb{R})Z_n^+(\mathbb{R})$ and $Z_n^+(\mathbb{R})$ is the subgroup of matrices of positive determinant in the center $Z_n(\mathbb{R})$ of $GL_n(\mathbb{R})$. For a given weight μ the set of representations π with this property is usually denoted by $\text{Coh}(\mu)$. The treatment of the critical values for $m = n - 1$ relied on the bijection $t \mapsto t + \frac{1}{2}$ in this case between the set $\text{Crit}(\pi_\infty, \sigma_\infty)$ of critical numbers and the parameter set $\text{Emb}(\nu, \check{\mu})$ of integers s allowing to embed the twists $M_{\nu-s} = \det^s \otimes M_\nu$ for $\nu - s = (\nu_1 - s, \dots, \nu_m - s)$ into the contragredient $M_{\check{\mu}}$ of M_μ considered as a GL_m -module under the restricted action of $GL_m \hookrightarrow GL_n$ where we had to

assume the existence of at least one such embeddable twist. Eventually this enabled us to interpret the critical values of $L(\pi, \sigma, s)$ as resulting from a pairing of appropriate cohomology spaces with coefficients in M_μ and M_ν thus supplying a powerful technique towards algebraicity and integrality statements for the critical values.

Unfortunately this approach seems not to work in general. Even in the case $m = n - 1$ it may happen that $\text{Emb}(\nu, \check{\mu})$ is empty although critical numbers do exist. In this article I will therefore by-pass this obstacle by not only restricting the GL_n -action on M_μ to a suitable smaller subgroup but also restricting the GL_m -action on M_ν and carefully analyze the correlation between certain irreducible components $M_{\theta'_i(\check{\lambda})}$, $M_{\theta_i(\check{\mu})}$ for $i = 1, 2$ of modifications of such restrictions to certain subgroups GL_r and GL_{r+1} . Like in [7] a technical tool is the ramification law for the restriction of an irreducible GL_{r+1} -representation M_α of highest weight α to GL_r saying that this restriction decomposes as a direct sum of irreducible GL_r -representations M_β of highest weight β each of multiplicity one where β varies subject to the condition

$$\alpha_\rho \geq \beta_\rho \geq \alpha_{\rho+1} \quad \text{for } \rho = 1, \dots, r.$$

We will write $\beta \prec \alpha$ in this case and we denote in general by

$$\text{Emb}(\beta, \alpha) := \{s \in \mathbb{Z}; \beta - s \prec \alpha\}$$

the parameter set of embeddable twists. The involved modifications in particular incorporate the relative position of the respective Langlands parameters (w, l) of π_∞ and (w', l') of σ_∞ via a certain *position tuple* $a \in \mathbb{Z}^m$ assuming without loss of generality that $l_1 > l'_1$. The components of a are uniquely determined by the requirement

$$l_{a_j} > l'_j \geq l_{1+a_j} \quad \text{for } j = 1, \dots, m$$

and in a first step the dominant weight ν is replaced by the modified weight λ where we put $\lambda_j := \nu_j + a_j - j$. In the *non-exceptional* case where n and m are not both odd λ turns out to be a dominant weight with purity property. In the exceptional case a slight further modification leads to these qualities.

To begin with in Proposition 2.1 we characterize the critical numbers by the inequality

$$|t - \kappa| < L + \frac{1}{2}$$

for $\kappa := \frac{1}{2}(w + w' + 1)$ and

$$L := \frac{1}{2} \min\{|l_i - l'_j|; \text{ all } i, j \text{ such that } i \neq \frac{n+1}{2} \text{ or } j \neq \frac{m+1}{2}\}$$

with an additional parity condition in the *exceptional* case when n and m both are odd. Then we discuss the so-called *jump indices* j where the position sequence a increases in the sense that $a_{j+1} > a_j$. A simple transformation rule relating the Langlands parameters (w, l) and (w', l') of π_∞ and σ_∞ with the corresponding weights μ and ν allows us to translate the characterizing inequality for critical numbers t into a system of inequalities of the form

$$(I_j) \quad \check{\mu}_{a_j} \geq \lambda_j - s \geq \check{\mu}_{1+a_j} \quad \text{for } j = 1, \dots, m$$

where $s = t + \frac{n-m}{2} - 1$ with slight modifications in the exceptional case (see Proposition 3.2). In terms of the jump indices j_1, \dots, j_k this inequality system can be expressed in the form

$$(S_\kappa) \quad \check{\mu}_{a_{j_\kappa}} \geq \lambda_{1+j_{\kappa-1}} - s \geq \dots \geq \lambda_{j_\kappa} - s \geq \check{\mu}_{1+a_{j_\kappa}}$$

for $\kappa = 1, \dots, k+1$ with $j_0 = 0$ and $j_{k+1} = m$. In fact by the dominance property of λ the inequality sequence (S_κ) is equivalent to demanding just the first and the last inequality. This observation is vital for us since it allows by a careful book-keeping to translate the inequality system via the ramification law into an embedding statement of GL_r -representations for $r := k+1$. This we work out first for a general system of inequalities in Proposition 4.1 which we then use as a guideline for the composition of suitably modified highest weights $\theta'_i(\tilde{\lambda})$ for GL_r and $\theta_i(\tilde{\mu})$ for GL_{r+1} with purity property for $i = 1, 2$. Eventually our main result in Theorem A (in the non-exceptional case) and Theorem B (in the exceptional case) unconditionally parametrizes the critical numbers as simultaneous parameters of embeddings that belong to $\text{Emb}(\theta'_i(\tilde{\lambda}), \theta_i(\tilde{\mu}))$ for $i = 1, 2$ thus generalizing the specific approach in the case $m = n-1$. In contrast to that special case in this paper the representation π a priori does not play a privileged role compared to σ . This fact is taken into account by the more symmetric formulation of the results in the two corollaries attached to the theorems where the critical numbers essentially get parametrized by the common 1-dimensional pieces of the canonically associated GL_r -representations

$$(M_{\theta_i(\tilde{\mu})} \otimes M_{\theta'_i(\tilde{\lambda})})^{SL_r} \quad \text{for } i = 1, 2$$

with pieces given by the *Tate modules* $T_r(s) := M_{(s, \dots, s)} = \det^s$. The very special case $n = m = 1$ is excluded here since the critical numbers of Dirichlet- L -functions are well understood.

One might interpret the results in this paper as a vague hint towards a possibly extended cohomological construction of *modular symbols* attached to the critical values of the L -function $L(\pi, \sigma, s)$ for cohomological cuspidal representations π and σ in general.

2 Langlands parameters of critical pairs

As is well known by (3.6) in [8] following Lemme 3.14 in [1] the infinity component π_∞ of a cohomological representation π of GL_n is an induced representation of Langlands type of the form

$$\pi_\infty \cong J(-w, l) \otimes \text{sgn}^\delta$$

(see also (1.8) in [7]), where the parameters $w \in \mathbb{Z}$ and $l = (l_i) \in \mathbb{Z}^n$ belong to the set

$$L_0^+(n) = \{(w, l) \in \mathbb{Z} \times \mathbb{Z}^n; l_i > l_{i+1}, l_i + l_{n+1-i} = 0, w + l_i \equiv n+1 \pmod{2}\}$$

and $\delta \in \{0, 1\}$. Similarly for a cohomological representation σ of GL_m the infinity component σ_∞ is of the form

$$\sigma_\infty \cong J(-w', l') \otimes \text{sgn}^{\delta'}$$

with $(w', l') \in L_0^+(m)$, where we use the same sign convention for w and w' as in [7](1.8).

In analogy with Deligne's notion of critical values of motivic L-functions we call a number $t \in \frac{m+n}{2} + \mathbb{Z}$ a *critical number* if the archimedean L-function $L(\pi_\infty, \sigma_\infty, s)$ and its counterpart $L(\check{\pi}_\infty, \check{\sigma}_\infty, 1-s)$ in the functional equation with the contragredients $\check{\pi}_\infty$ and $\check{\sigma}_\infty$ do not have a pole in t . Recall that the L-function $L(\pi_\infty, \sigma_\infty, s)$ is defined to be the L-function of the representation $\tau := \pi_\infty^W \otimes \sigma_\infty^W$ of the Weil group $W_\mathbb{R}$ where π_∞^W and σ_∞^W denote the semisimple representations of $W_\mathbb{R}$ attached to π_∞ and σ_∞ by the archimedean local Langlands correspondence. We denote by $\text{Crit} = \text{Crit}(\pi_\infty, \sigma_\infty)$ the set of critical numbers t and we call the pair $(\pi_\infty, \sigma_\infty)$ a *critical pair*, if Crit is not empty. This finite set can be described in terms of the associated Langlands parameters as follows. We put

$$L_0 := \min\{|l_i - l'_j|; \text{ all pairs } (i, j) \text{ such that } i \neq \frac{n+1}{2} \text{ or } j \neq \frac{m+1}{2}\}$$

we set $L := L_0/2$. Note that the restriction for the choice of i and j under consideration in the minimum only matters in the exceptional case $n \equiv m \equiv 1 \pmod{2}$. Otherwise i and j vary unconditionally. Let $\kappa := \frac{1}{2}(w + w' + 1)$ and $\kappa' := \kappa - \frac{1}{2}$.

Proposition 2.1: *A number $t \in \frac{m+n}{2} + \mathbb{Z}$ is critical if and only if*

$$(I) \quad |t - \kappa| < L + \frac{1}{2}$$

and in addition in case $n \equiv m \equiv 1 \pmod{2}$ the number t satisfies the parity condition

$$(PC)_\epsilon \quad t - \kappa' \in (2\mathbb{N} - \epsilon) \cup -(2\mathbb{N} - 1 - \epsilon)$$

for $\epsilon = 0, 1$ with $\epsilon \equiv \delta + \delta' \pmod{2}$.

Proof. In the special case $m = n - 1$ in [7] a proof has been worked out which we now want to extend to the general case. The semisimple representation π_∞^W of the Weil group $W_\mathbb{R}$ attached to π_∞ is of the form

$$\pi_\infty^W = (l_1, -w/2) \oplus \dots \oplus (l_{n/2}, -w/2)$$

for even n and

$$\pi_\infty^W = (l_1, -w/2) \oplus \dots \oplus (l_{\frac{n-1}{2}}, -w/2) \oplus (sgn^\delta, -w/2)$$

for odd n , where (l, t) and (\pm, t) for integers $l \geq 1$ and $t \in \mathbb{C}$ are the irreducible 2-dimensional resp. 1-dimensional representations of $W_\mathbb{R}$ in Knapp's notation ([6], (3.2), (3.3)). Analogously we decompose σ_∞^W in terms of the parameters l'_j , w' and δ' . The L-function of $\tau = \pi_\infty^W \otimes \sigma_\infty^W$ is given by the product of the L-functions $L(s, \varphi)$ of the irreducible constituents φ of τ . Using Deligne's terminology $\Gamma_\mathbb{R}(s) := \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_\mathbb{C}(s) := 2(2\pi)^{-s}\Gamma(s)$ we can express the $L(s, \varphi)$ by (3.6) in [6] as

$$(2.1) \quad L(s, (l, t)) = \Gamma_\mathbb{C}(s + t + \frac{l}{2})$$

and

$$(2.2) \quad L(s, (sgn^\epsilon, t)) = \Gamma_{\mathbb{R}}(s + t + \epsilon)$$

with $\epsilon = 0, 1$.

In order to write down the L-function for τ (and the contragredient $\check{\tau}$) we must work out the decomposition into irreducible parts. To simplify notation we will sometimes use the reducible representations $(0, t) := (+, t) \oplus (-, t)$. There are three possible parity combinations for n and m to be treated separately.

Lemma 2.1: *For $n \equiv m \equiv 0 \pmod{2}$ we have*

$$\tau = \bigoplus_{i=1}^{n/2} \bigoplus_{j=1}^{m/2} (l_i + l'_j, -\kappa') \oplus (|l_i - l'_j|, -\kappa')$$

hence

$$L(s, \tau) = \prod_{i,j} \Gamma_{\mathbb{C}}(s - \kappa' + \frac{l_i + l'_j}{2}) \cdot \prod_{i,j; l_i \neq l'_j} \Gamma_{\mathbb{C}}(s - \kappa' + \frac{|l_i - l'_j|}{2}) \cdot \prod_{i,j; l_i = l'_j} \Gamma_{\mathbb{R}}(s - \kappa') \Gamma_{\mathbb{R}}(s - \kappa' + 1).$$

The proof of Lemma 2.1 is an easy exercise using the rules for the tensor product of the involved irreducible representations

$$(sgn^\epsilon, t) \otimes (sgn^{\epsilon'}, t') = (sgn^{\epsilon+\epsilon'}, t + t'),$$

$$(l, t) \otimes (sgn^\epsilon, t) = (l, t + t'),$$

$$(l, t) \otimes (l', t') = (l + l', t + t') \oplus (|l - l'|, t + t')$$

as dicussed for instance in [7] p.214. The formula for the L-function hereafter follows by (2.1) and (2.2).

Lemma 2.2: *For $n \equiv m + 1 \equiv 0 \pmod{2}$ we get*

$$\tau = \bigoplus_{i=1}^{n/2} \bigoplus_{j=1}^{(m-1)/2} (l_i + l'_j, -\kappa') \oplus (|l_i - l'_j|, -\kappa') \oplus \bigoplus_{i=1}^{n/2} (l_i, -\kappa')$$

hence

$$L(s, \tau) = \prod_{i,j} \Gamma_{\mathbb{C}}(s - \kappa' + \frac{l_i + l'_j}{2}) \cdot \prod_{i,j; l_i \neq l'_j} \Gamma_{\mathbb{C}}(s - \kappa' + \frac{|l_i - l'_j|}{2}) \cdot \prod_{i,j; l_i = l'_j} \Gamma_{\mathbb{R}}(s - \kappa') \Gamma_{\mathbb{R}}(s - \kappa' + 1) \cdot \prod_{i=1}^{n/2} \Gamma_{\mathbb{C}}(s - \kappa' + \frac{l_i}{2}).$$

The proof of Lemma 2.2 and the proof of the following Lemma 2.3 are very much similar to the proof of Lemma 2.1 so we omit it.

The remaining so-called exceptional case $m \equiv n \equiv 1 \pmod{2}$ will play a particular role later-on. We put $\epsilon = 0$ or 1 such that $\epsilon \equiv \delta + \delta' \pmod{2}$.

Lemma 2.3: *In the exceptional case we find*

$$\begin{aligned} \tau = & \bigoplus_{i=1}^{(n-1)/2} \bigoplus_{j=1}^{(m-1)/2} (l_i + l'_j, -\kappa') \oplus (|l_i - l'_j|, -\kappa') \\ & \oplus \bigoplus_{i=1}^{(n-1)/2} (l_i, -\kappa') \oplus \bigoplus_{j=1}^{(m-1)/2} (l'_j, -\kappa') \oplus (\text{sgn}^\epsilon, -\kappa'), \end{aligned}$$

hence

$$\begin{aligned} L(s, \tau) = & \prod_{i,j} \Gamma_{\mathbb{C}}(s - \kappa' + \frac{l_i + l'_j}{2}) \cdot \prod_{i,j; l_i \neq l'_j} \Gamma_{\mathbb{C}}(s - \kappa' + \frac{|l_i - l'_j|}{2}) \\ & \cdot \prod_{i,j; l_i = l'_j} \Gamma_{\mathbb{R}}(s - \kappa') \Gamma_{\mathbb{R}}(s - \kappa' + 1) \\ & \cdot \prod_{i=1}^{(n-1)/2} \Gamma_{\mathbb{C}}(s - \kappa' + \frac{l_i}{2}) \cdot \prod_{j=1}^{(m-1)/2} \Gamma_{\mathbb{C}}(s - \kappa' + \frac{l'_j}{2}) \cdot \Gamma_{\mathbb{R}}(s - \kappa' + \epsilon). \end{aligned}$$

Remark 2.1: *In all cases the L -function $L(s, \tilde{\tau})$ of the contragredient representation is given by the same formulas as for $L(s, \tau)$ but with κ' replaced by $-\kappa'$, since $\tilde{\pi}_\infty \cong J(w, l) \otimes \text{sgn}^\delta$ and $\tilde{\sigma}_\infty \cong J(w', l') \otimes \text{sgn}^{\delta'}$.*

Next we want to get rid of the unpleasant Γ -factors attached to the pairs (i, j) where $l_i = l'_j$.

Lemma 2.4: *If there is a pair (i, j) such that $l_i = l'_j \neq 0$, then for all $t \in \frac{n+m}{2} + \mathbb{Z}$ either t is a pole of $\Gamma_{\mathbb{R}}(s - \kappa') \Gamma_{\mathbb{R}}(s - \kappa' + 1)$ or t is a pole of $\Gamma_{\mathbb{R}}(1 - s + \kappa') \Gamma_{\mathbb{R}}(1 - s + \kappa' + 1)$, hence t is a pole of $L(s, \tau)$ or of $L(1 - s, \tilde{\tau})$. In particular Crit is empty in this case.*

Proof of Lemma 2.4. The set of poles of the first product is $\kappa' + \mathbb{Z}_{\leq 0}$ and that of the second product is $\kappa' + \mathbb{N}$, hence each number of the form $\kappa' + z$ with an integer z occurs as a pole. Now the assumption $l_i = l'_j$ implies by means of the congruence properties $w + l_i \equiv n + 1 \pmod{2}$ and $w' + l'_j \equiv m + 1 \pmod{2}$ that $w + w' \equiv n + m \pmod{2}$ hence $\kappa' + \mathbb{Z} = \frac{n+m}{2} + \mathbb{Z}$ and therefore any t in this set is a pole which completes the proof.

Now once we assume the existence of a critical number $t \in \frac{n+m}{2} + \mathbb{Z}$ the unpleasant Γ -factors of $L(s, \tau)$ attached to pairs (i, j) with $l_i = l'_j (\neq 0)$ in the Lemmas 2.1, 2.3 cannot turn up by Lemma 2.4. The regularity condition for the remaining explicit Γ -factors of $L(s, \tau)$ and $L(1 - s, \tilde{\tau})$ says $|t - \kappa| < L_1 + 1/2$ for

$$L_1 := \frac{1}{2} \min_{i=1}^{n/2} \min_{j=1}^{m/2} \{l_i + l'_j, |l_i - l'_j|\}$$

in the case $n \equiv m \equiv 0 \pmod{2}$, $|t - \kappa| < L_2 + 1/2$ for

$$L_2 := \frac{1}{2} \min_{i=1}^{n/2} \min_{j=1}^{(m-1)/2} \{l_i + l'_j, |l_i - l'_j|, l_i, l'_j\}$$

in the case $n \equiv m + 1 \equiv 0 \pmod{2}$, and eventually $|t - \kappa| < L_3 + 1/2$ for

$$L_3 := \frac{1}{2} \min_{i=1}^{(n-1)/2} \min_{j=1}^{(m-1)/2} \{l_i + l'_j, |l_i - l'_j|, l_i, l'_j\}$$

in the exceptional case $n \equiv m \equiv 1 \pmod{2}$, where in addition taking in account the last factor $\Gamma_{\mathbb{R}}(s - \kappa' + \epsilon)$ of $L(s, \tau)$ we encounter the extra condition that t be not a pole of $\Gamma_{\mathbb{R}}(s - \kappa' + \epsilon)\Gamma_{\mathbb{R}}(1 - s + \kappa' + \epsilon)$. Since we know the poles of $\Gamma(s)$ this is equivalent to the parity condition

$$(PC_{\epsilon}) \quad t - \kappa' \in (2\mathbb{N} - \epsilon) \cup -(2\mathbb{N} - 1 - \epsilon)$$

in Proposition 2.1.

Eventually we simplify and unify the bounds L_i ($i=1,2,3$) by showing that they are in fact all equal to the bound L in the proposition. This is obvious for L_1 since

$$|\pm l_i \pm l'_j| \leq l_i + l'_j$$

for all positive l_i, l'_j and hence also for

$$L_2 = \frac{1}{2} \min_{i=1}^n \min_{j=1}^m \{|l_i - l'_j|\} = L,$$

where for $j = (m + 1)/2$ we have $|l_i - l'_j| = |l_i|$. The same observation applies to $L_3 = L$.

In the opposite direction of the proposition for a number $t \in \frac{n+m}{2} + \mathbb{Z}$ we have $|t - \kappa| \geq 1/2$ or $t = \kappa$, hence (I) implies $L > 0$ in the first case. In the second case $\frac{n+m}{2} + \kappa$ is integral, i.e. $n + m \equiv w + w' + 1 \pmod{2}$, hence by the congruences $w + l_i \equiv n + 1 \pmod{2}$ and $w' + l'_j \equiv m + 1 \pmod{2}$ we get $l_i + l'_j \equiv n + m + w + w' \equiv 1 \pmod{2}$ and in particular $l_i \neq l'_j$ for all i, j , so $L > 0$ in this case too. Therefore the L-function has no unpleasant Γ -factors like those in Lemma 2.4 and t satisfying (I) and $(PC)_{\epsilon}$ in addition in the exceptional case is a critical number. This completes the proof of the proposition.

Remark 2.2: *There is an obvious reflection map*

$$\theta : \text{Crit} \longrightarrow \text{Crit}, t \mapsto w + w' + 1 - t.$$

Proposition 2.2: *If the set Crit is non empty then for $i = 1, \dots, n$ and $j = 1, \dots, m$ we have for each pair (i, j) the alternative*

$$l_i \neq l'_j \text{ or } l_i = l'_j = 0.$$

The second case only occurs for $n \equiv m \equiv 1 \pmod{2}$ with $i = \frac{n+1}{2}, j = \frac{m+1}{2}$.

Proof. The existence of a critical t implies $L \neq 0$ hence in particular $L_0 \neq 0$, so $l_i \neq l'_j$ for $i \leq \frac{n}{2}$ and $j \leq \frac{m}{2}$ which are the positive components of l and l' .

Therefore certainly we have $l_i \neq -l'_j = l'_{m+1-j}$ and $l'_j \neq -l_i = l_{n+1-i}$. So the only remaining case is $l_i = l'_j = 0$ as described.

In the non-exceptional case we can reverse the conclusion in proposition 2.2.

Proposition 2.3: *If we are not in the exceptional case $n \equiv m \equiv 1 \pmod{2}$ the set of critical numbers is non empty if and only if for each pair (i, j) we have $l_i \neq l'_j$, i.e. $L_0 \neq 0$. In this situation the two numbers $t_0 := L - 1 + \frac{w+w'+1}{2}$ and $\theta(t_0)$ are critical.*

Proof. We are left to show the existence of a critical number if L_0 does not vanish. Now for $L_0 \neq 0$ obviously t_0 is critical.

From now on we will assume the existence of a critical number. Since we excluded the case $n = m = 1$ in this paper we may and will assume that the Langlands parameters satisfy the

Hypothesis: $l_1 > l'_1$.

The correlation of the Langlands parameters l and l' is recorded by the *position tuple* of integers $a = (a_1, \dots, a_m)$ uniquely determined by the property

$$l_{a_j} > l'_j \geq l_{1+a_j} \text{ for } j = 1, \dots, m.$$

Obviously we have

$$1 \leq a_1 \leq a_2 \leq \dots \leq a_m \leq n-1.$$

In general the sequence a_j is not constant.

Lemma 2.5: *The position tuple a enjoys the symmetry*

$$(2.3) \quad a_j + a_{m+1-j} = n \text{ for } j = 1, \dots, m$$

except for $j = \frac{m+1}{2}$ in the exceptional case $m \equiv n \equiv 1 \pmod{2}$ where $a_j = \frac{n-1}{2}$.

This follows immediately from the defining inequalities via the symmetry of the Langlands parameters

$$l_i + l_{n+1-i} = 0 \text{ and } l'_j + l'_{m+1-j} = 0$$

Remark 2.3: *The constant case: If $m > 1$ and $a_1 = a_m$ necessarily n must be even, $a_i = n/2$ and*

$$l_{\frac{n}{2}} > l'_1 > l'_2 > \dots > l'_m > -l_{\frac{n}{2}}.$$

Assuming that n is even we arrive at the same conclusions for $m = 1$. For odd n and $m = 1$ we get $a_1 = \frac{n-1}{2}$.

If a is not constant there are *jump indices*. We call j a *jump index* if we have $a_j < a_{j+1}$. Let $\{j_1, \dots, j_k\}$ denote the set of *jump indices* and let $k := 0$ for

constant a . Note that for $m = 1$ certainly a is constant hence $k = 0$. Suppose we have $m > 1$. Then by the symmetry property (2.3) we get

Remark 2.4: *In the non-exceptional case j is a jump index if and only if $m - j$ is a jump index. In the exceptional case this holds for $j \neq \frac{m+1}{2}$ and moreover $j = \frac{m+1}{2}$ is always a jump index whereas $\frac{m-1}{2}$ is a jump index if and only if we have $l_{\frac{n-1}{2}} < l'_{\frac{m-1}{2}}$, which therefore is equivalent to k being even. Furthermore except for the exceptional case with odd k the jump indices enjoy the symmetry property*

$$(2.4) \quad j_\kappa + j_{k+1-\kappa} = m \text{ for } \kappa = 1, \dots, k.$$

In the excluded case this symmetry still holds for $\kappa \neq \frac{k+1}{2}$ whereas we have $j_{\frac{k+1}{2}} = \frac{m+1}{2}$.

In terms of the position tuple a and we can express in the non-exceptional case the bound L in Proposition 2.1 as

$$L = \frac{1}{2} \min\{l_{a_j} - l'_j, l'_j - l_{1+a_j}; j = 1, \dots, m\}$$

hence the inequality (I) in Proposition 2.1 for (and so the characterization of) critical numbers transforms into the simultaneous inequalities

$$\pm(t - \frac{w + w' - 1}{2}) < \frac{1}{2}(l_{a_j} - l'_j + 1), \frac{1}{2}(l'_j - l_{1+a_j} + 1)$$

for $j = 1, \dots, m$. In the exceptional case we get

$$L = \frac{1}{2} \min\{l'_{\frac{m-1}{2}}, l_{\frac{n-1}{2}}, l_{a_j} - l'_j, l'_j - l_{1+a_j}; j \neq \frac{m+1}{2}\}$$

and the respective modification of the inequalities for critical numbers.

3 Highest weights and critical numbers

For arbitrary n we consider irreducible algebraic representations (ρ_μ, M_μ) of GL_n over \mathbb{Q} with dominant highest weight μ , i.e. $\mu = (\mu_1, \dots, \mu_n)$ in

$$X^+(n) := \{\mu \in \mathbb{Z}^n; \mu_1 \geq \mu_2 \geq \dots \geq \mu_n\}.$$

We are in particular interested in the essentially selfdual representations. These are precisely those representations which satisfy the *purity condition* saying that for a suitable integer $w = wt(\mu)$ the so called *weight* of μ we have

$$\mu_i + \mu_{n+1-i} = w \text{ for all } i.$$

Recall that by (3.5) and (3.6) in [8] the set $X_0^+(n)$ of these μ corresponds to the set of Langlands parameters $L_0^+(n)$ via the bijection

$$L_0^+(n) \longrightarrow X_0^+(n)$$

$$(w, l) \mapsto \mu = \left(\frac{w + l_1 + 1 - n}{2}, \frac{w + l_2 + 3 - n}{2}, \dots, \frac{w + l_n + n - 1}{2} \right),$$

where the inverse mapping is given by $\mu \mapsto (w, l)$ with $w = \mu_1 + \mu_n$ and $l_i = 2\mu_i + n + 1 - w - 2i$, and where any $\pi \in \text{Coh}(\mu)$ has infinity component

$$\pi_\infty \cong J(-w, l) \otimes \text{sgn}^\delta$$

for suitable δ . Via this bijection we want to express the previous simultaneous inequalities for critical numbers in terms of the associated highest weights. So let $\mu \in X_0^+(n)$ and $\nu \in X_0^+(m)$ correspond to the Langlands parameters (w, l) and (w', l') as considered in the previous section for $(\pi, \sigma) \in \text{Coh}(\mu) \times \text{Coh}(\nu)$ with a critical pair $\pi_\infty, \sigma_\infty$. Using the position tuple a we introduce the weight vector $\lambda = (\lambda_1, \dots, \lambda_m)$ where we put

$$\lambda_j := \nu_j + a_j - j \text{ for } j = 1, \dots, m.$$

Proposition 3.1: *The weight λ is dominant and satisfies the purity condition*

$$(P_j) \quad \lambda_j + \lambda_{m+1-j} = w' + n - m - 1$$

for $j \neq \frac{m+1}{2}$. In the exceptional case $n \equiv m \equiv 1 \pmod{2}$ we have

$$2\lambda_{\frac{m+1}{2}} = w' + n - m - 2,$$

whereas in the non-exceptional case (P_j) holds for all j .

An obvious consequence is

Corollary 3.1: *In the non-exceptional case we have $\lambda \in X_0^+(m)$. In the exceptional case for $m > 1$ we may modify λ to achieve the purity property for*

$$\lambda_{\text{mod}} := (\lambda_1, \dots, \lambda_{\frac{m+1}{2}}, \lambda_{\frac{m+3}{2}} - 1, \dots, \lambda_m - 1) \in X_0^+(m)$$

and the truncated

$$\lambda_{\text{tr}} := (\lambda_1, \dots, \lambda_{\frac{m-1}{2}}, \lambda_{\frac{m+3}{2}}, \dots, \lambda_m) \in X_0^+(m-1).$$

For $m = 1$ always $\lambda \in X_0^+(1)$.

Proof. If j is not a jump index we have $a_j = a_{j+1}$ hence

$$\lambda_j = \nu_j + a_j - j \geq \nu_{j+1} + a_{j+1} - j > \lambda_{j+1}$$

since ν is dominant. Otherwise we have $a_j + 1 \leq a_{j+1}$ so the inequalities defining the position tuple a in particular imply that we have $l'_j \geq l'_{1+a_j}$ and $l'_{a_{j+1}} > l'_{j+1}$. In terms of the associated highest weights this says

$$2\nu_j + m + 1 - w' - 2j \geq 2\mu_{1+a_j} + n + 1 - w - 2(1 + a_j)$$

and

$$2\mu_{a_{j+1}} + n + 1 - w - 2a_{j+1} > 2\nu_{j+1} + m + 1 - w' - 2(j + 1).$$

So with $R := \frac{1}{2}(m - n + w - w')$ we get

$$\lambda_j + R \geq \mu_{1+a_j} - 1 \text{ and } \mu_{a_{j+1}} > \lambda_{j+1} + R.$$

Since $\mu_{1+a_j} \geq \mu_{a_{j+1}}$ this implies $\lambda_j \geq \lambda_{j+1}$ which proves that λ is dominant.

In the non-exceptional case the purity property (P_j) follows directly from the purity of ν and the symmetry property (AS) of the position tuple a . In the exceptional case the same argument works for $j \neq \frac{m+1}{2}$ and for $j = \frac{m+1}{2}$ the purity of ν and the identity $a_{\frac{m+1}{2}} = \frac{n-1}{2}$ supply the claimed extra formula for $\lambda_{\frac{m+1}{2}}$.

We now turn to the *dual* highest weight $\check{\mu}$ with the components $\check{\mu}_j = -\mu_{n+1-j}$ and corresponding Langlands parameter $(-w, l)$.

Proposition 3.2: *In the non-exceptional case a number $t \in \frac{m+n}{2} + \mathbb{Z}$ satisfies the inequality (I) hence is critical if and only if the integer $s := t + \frac{n-m}{2} - 1$ satisfies the simultaneous inequalities*

$$(I_j) \quad \check{\mu}_{a_j} \geq \lambda_j - s \geq \check{\mu}_{1+a_j} \text{ for } j = 1, \dots, m.$$

In the exceptional case the condition is that s fulfills (I_j) for all $j \neq \frac{m+1}{2}$ and the “middle condition”

$$\check{\mu}_{\frac{n-1}{2}} \geq \lambda_{\frac{m+1}{2}} - s \geq \check{\mu}_{\frac{n+3}{2}} - 1 \text{ for even } k,$$

respectively

$$\lambda_{\frac{m-1}{2}} \geq \check{\mu}_{\frac{n+1}{2}} + s \geq \lambda_{\frac{m+3}{2}} \text{ for odd } k.$$

Proof. As we observed at the end of the previous section a number t satisfying (I) is characterized by a system of inequalities involving certain Langlands parameters l_i and l'_j which we now express in terms of the associated weights

$$l_i = 2\check{\mu}_i + n + 1 + w - 2i$$

respectively

$$l'_j = 2\nu_j + m + 1 - w' - 2j.$$

In the non-exceptional case this system gets the form

$$\pm(t - \frac{w + w' + 1}{2}) < \check{\mu}_{a_j} - \nu_j - a_j + j + \frac{n - m}{2} + \frac{w + w' + 1}{2}$$

and

$$\pm(t - \frac{w + w' + 1}{2}) < \nu_j + a_j - j - \check{\mu}_{1+a_j} - \frac{n - m}{2} - \frac{w + w' + 1}{2} + 2$$

for $j = 1, \dots, m$. In the exceptional case the system consists of these inequalities for $j \neq \frac{m+1}{2}$ and the inequality

$$\pm(t - \frac{w + w' + 1}{2}) < \check{\mu}_{\frac{n-1}{2}} + \frac{3 + w}{2} \text{ for even } k$$

respectively

$$\pm(t - \frac{w + w' + 1}{2}) < \nu_{\frac{m-1}{2}} + \frac{3 - w'}{2} \text{ for odd } k.$$

By the purity property of ν and $\check{\mu}$ our system of inequalities reduces to the simultaneous inequalities (I_j) and the last inequality transforms into the “middle condition” which completes the proof.

Remark 3.1: *In the exceptional case for odd k the inequalities $(I_{\frac{m-1}{2}})$ and $(I_{\frac{m+3}{2}})$ imply the “middle condition” hence the latter can be deleted in the criterion of the proposition.*

Taking into account the jump indices j_κ of the position tuple a we can easily reformulate Proposition 3.2. Set $j_0 := 0$ and $j_{k+1} := m$.

Proposition 3.3: *In the non-exceptional case a number $t \in \frac{m+n}{2} + \mathbb{Z}$ satisfies the inequality (I) (hence is critical) if and only if the integer $s := t + \frac{n-m}{2} - 1$ satisfies the simultaneous inequalities*

$$(S_\kappa) \quad \check{\mu}_{a_{j_\kappa}} \geq \lambda_{1+j_{\kappa-1}} - s \geq \dots \geq \lambda_{j_\kappa} - s \geq \check{\mu}_{1+a_{j_\kappa}}$$

for $\kappa = 1, \dots, k+1$. Furthermore the inequalities of the listed λ -components are strict, i.e.

$$\lambda_{1+j_{\kappa-1}} > \lambda_{2+j_{\kappa-1}} > \dots > \lambda_{j_\kappa}.$$

In the exceptional case (I) holds for odd k if and only if the integer s satisfies (S_κ) for all $\kappa \neq \frac{k+1}{2}$ and for $\kappa = \frac{k+1}{2}$ the truncated series of inequalities

$$(S'_{\frac{k+1}{2}}) \quad \check{\mu}_{\frac{n-1}{2}} \geq \lambda_{1+j_{\frac{k-1}{2}}} - s \geq \dots \geq \lambda_{j_{\frac{k+1}{2}}-1} - s \geq \check{\mu}_{\frac{n+1}{2}}$$

where $j_{\frac{k+1}{2}} = \frac{m+1}{2}$ and $a_{\frac{m+1}{2}} = a_{\frac{m-1}{2}} = \frac{n-1}{2}$. For even k the integer s has to satisfy (S_κ) for all $\kappa \neq 1 + \frac{k}{2}$ and the “middle condition”

$$\check{\mu}_{\frac{n-1}{2}} \geq \lambda_{\frac{m+1}{2}} - s \geq \check{\mu}_{\frac{n+3}{2}} - 1.$$

4 Inequality systems and ramification

Our goal in this section is to interpret via the ramification law of restricted irreducible representations the simultaneous inequalities in Proposition 3.3 as embeddings of representations attached to suitable choices of highest weights. To begin with we observe that the essential part of the inequality sequence (S_κ) is the equivalent four term inequality

$$(4.1) \quad \check{\mu}_{a_{j_\kappa}} \geq \lambda_{1+j_{\kappa-1}} - s \geq \lambda_{j_\kappa} - s \geq \check{\mu}_{1+a_{j_\kappa}}.$$

With that in mind we are led to study arbitrary systems of inequalities of the form

$$(4.2) \quad u_{2\rho-1} \geq v_{2\rho-1} \geq v_{2\rho} \geq u_{2\rho} \text{ for } \rho = 1, \dots, r$$

where $u = (u_1, \dots, u_{2r})$ and $v = (v_1, \dots, v_{2r})$ are dominant weights for GL_{2r} with purity property, i.e. $u, v \in X_0^+(2r)$. In this situation we say that the pair (u, v) of highest weights satisfies (4.2). In the total $4r$ -term inequality sequence (4.2) we must take care of the *middle group*

$$v_r \geq u_r \geq u_{r+1} \geq v_{r+1} \quad \text{for even } r,$$

respectively

$$u_r \geq v_r \geq v_{r+1} \geq u_{r+1} \quad \text{for odd } r.$$

For a highest weight $y \in X_0^+(2r)$ we call the difference of the middle components $d(y) := y_r - y_{r+1}$ the *defect* of y .

Remark 4.1: For a pair (u, v) of $u, v \in X_0^+(2r)$ satisfying (4.2) we necessarily have $d(u) \leq d(v)$ for even r and $d(v) \leq d(u)$ for odd r .

For technical reasons we suppose for the time being that the middle group is *special* in the sense that $d(u) = 0$ or $d(v) = 0$. For the set of such special highest weights we introduce the notation

$$X_{sp}(2r) := \{y \in X_0^+(2r); d(y) = 0\}.$$

We will consider “splitting maps” $\theta = (\theta_1, \theta_2), \theta' = (\theta'_1, \theta'_2)$ with

$$\theta : X_{sp}(2r) \longrightarrow X_0^+(r+1) \times X_0^+(r+1),$$

$$\theta' : X_0^+(2r) \longrightarrow X_0^+(r) \times X_0^+(r)$$

for even r and

$$\theta : X_0^+(2r) \longrightarrow X_0^+(r+1) \times X_0^+(r+1),$$

$$\theta' : X_{sp}(2r) \longrightarrow X_0^+(r) \times X_0^+(r)$$

for odd r defined as follows. For $y \in X_{sp}(2r)$ and $z \in X_0^+(2r)$ we put for even r

$$\theta_1(y) := (y_1, y_2, y_4, \dots, y_r = y_{r+1}, y_{r+3}, \dots, y_{2r-1}, y_{2r}),$$

$$\theta_2(y) := (y_1, y_3, \dots, y_{r-1}, y_{r+1} = y_r, y_{r+2}, y_{r+4}, \dots, y_{2r})$$

and

$$\theta'_1(z) := (z_2, z_4, \dots, z_r, z_{r+1}, z_{r+3}, \dots, z_{2r-1}),$$

$$\theta'_2(z) := (z_1, z_3, \dots, z_{r-1}, z_{r+2}, z_{r+4}, \dots, z_{2r}).$$

For odd r we similarly define

$$\theta_1(z) := (z_1, z_2, z_4, \dots, z_{r-1}, z_{r+2}, z_{r+4}, \dots, z_{2r-1}, z_{2r}),$$

$$\theta_2(z) := (z_1, z_3, \dots, z_r, z_{r+1}, z_{r+3}, \dots, z_{2r})$$

and

$$\theta'_1(y) := (y_2, y_4, \dots, y_{r+1} = y_r, y_{r+2}, y_{r+4}, \dots, y_{2r-1}),$$

$$\theta'_2(y) := (y_1, y_3, \dots, y_r = y_{r+1}, y_{r+3}, \dots, y_{2r}).$$

In particular for $r = 1$ where $y_1 = y_2$ we take $\theta_i(z) = z$ and $\theta'_i(y) = y_1$ for $i = 1, 2$.

Remark 4.2: Obviously the images of θ and θ' consist of dominant weights and they inherit the purity property of y and z .

Our main technical tool later-on will be

Proposition 4.1: For a given pair (y, z) of highest weights for GL_{2r} with $y \in X_{sp}(2r)$ and $z \in X_0^+(2r)$ the following statements are equivalent:

- a) The pair (y, z) satisfies (4.2) for even r respectively the pair (z, y) satisfies (4.2) for odd r .
- b) The images under the splitting maps θ and θ' have the embedding property

$$\theta'_i(z) \prec \theta_i(y) \quad (i = 1, 2) \quad \text{for even } r$$

respectively

$$\theta'_i(y) \prec \theta_i(z) \quad (i = 1, 2) \quad \text{for odd } r.$$

Proof. We first work out the proof for even r . The inequality system (4.2) for the pair (y, z) may be separated into two groups of inequalities. The first group is

$$\begin{aligned} (y_1 \geq) z_2 &\geq y_2, \\ (y_i \geq) z_{i+2} &\geq y_{i+2} \quad \text{for even } i, \quad 2 \leq i \leq r-2, \\ y_i &\geq z_i (\geq y_{i+2}) \quad \text{for odd } i, \quad r+1 \leq i \leq 2r-3, \\ y_{2r-1} &\geq z_{2r-1} (\geq y_{2r}). \end{aligned}$$

The second group is

$$\begin{aligned} y_i &\geq z_i (\geq y_{i+2}) \quad \text{for odd } i, \quad 1 \leq i \leq r-1, \\ (y_{i-2} \geq) z_i &\geq y_i \quad \text{for even } i, \quad r+2 \leq i \leq 2r. \end{aligned}$$

Now the first group precisely describes the embedding statement $\theta'_1(z) \prec \theta_1(y)$ and the second group expresses the statement $\theta'_2(z) \prec \theta_2(y)$, hence the statements a) and b) are equivalent.

For odd r the inequality system (4.2) for the pair (z, y) similarly separates into two groups. Now the first group is

$$\begin{aligned} (z_1 \geq) y_2 &\geq z_2, \\ (z_{i-2} \geq) y_i &\geq z_i \quad \text{for even } i, \quad 4 \leq i \leq r-1, \\ z_i &\geq y_i (\geq z_{i+2}) \quad \text{for odd } i, \quad r+2 \leq i \leq 2r-3, \\ z_{2r-1} &\geq y_{2r-1} (\geq z_{2r}). \end{aligned}$$

The second group is

$$\begin{aligned} z_i &\geq y_i (\geq z_{i+2}) \quad \text{for odd } i, \quad 1 \leq i \leq r, \\ (z_{i-2} \geq) y_i &\geq z_i \quad \text{for even } i, \quad r+1 \leq i \leq 2r. \end{aligned}$$

Here the first group describes the embedding statement $\theta'_1(y) \prec \theta_1(z)$ whereas the second group says $\theta'_2(y) \prec \theta_2(z)$, hence like in the even case the statements a) and b) are equivalent and the proof is complete.

We can easily extend the criterion of Proposition 4.1 to an arbitrary pair (u, v) of highest weights $u, v \in X_0^+(2r)$. Let $l := (0, \dots, 0, 1, \dots, 1) \in \mathbb{Z}^{2r}$ have a zero in each of the first r components and 1 in each of the remaining r components. Further let d denote the minimum of the defects $d(u), d(v)$, i.e.

$$d := \min\{u_r - u_{r+1}, v_r - v_{r+1}\},$$

and define the modified weights $\hat{u} := u + d \cdot l$ and $\hat{v} := v + d \cdot l$ such that in particular we have $d(\hat{u}) = 0$ or $d(\hat{v}) = 0$.

Corollary 4.2: *The pair (u, v) satisfies (4.2) if and only if $d(\hat{u}) = 0$ for even r respectively $d(\hat{v}) = 0$ for odd r and if we have the two embeddings*

$$\theta'_i(\hat{v}) \prec \theta_i(\hat{u}) \quad \text{for } i = 1, 2.$$

Proof. Certainly \hat{u} and \hat{v} still belong to $X_0^+(2r)$ and (u, v) satisfies (4.2) if and only if (\hat{u}, \hat{v}) does. Now we can apply Proposition 4.1 to (\hat{u}, \hat{v}) . If (4.2) holds for (\hat{u}, \hat{v}) then necessarily $d(\hat{u}) = 0$ for even r and $d(\hat{v}) = 0$ for odd r by Remark 4.1. In both cases we get $\theta'_i(\hat{v}) \prec \theta_i(\hat{u})$ for $i = 1, 2$ by Proposition 4.1. Conversely if we have $d(\hat{u}) = 0$ for even r respectively $d(\hat{v}) = 0$ for odd r and $\theta'_i(\hat{v}) \prec \theta_i(\hat{u})$ for $i = 1, 2$ then the proposition yields (4.2) for (\hat{u}, \hat{v}) in the even case and in the odd case as well.

5 Critical numbers and embeddings

We now want to apply the technique from the previous section in order to parametrize the set Crit of critical numbers of a critical pair $(\pi_\infty, \sigma_\infty)$ attached to $(\pi, \sigma) \in \text{Coh}(\mu) \times \text{Coh}(\nu)$. For this purpose we will use the system of simultaneous inequalities (S_κ) in Proposition 3.3 which we must reformulate slightly modified in a suitable form adjusted to the requirements of Corollary 4.2..

The non-exceptional case being less involved we first deal with this case. In view of Proposition 3.3 and (4.1) we put

$$u_{2\rho-1} := \check{\mu}_{a_{j_\rho}}, \quad u_{2\rho} := \check{\mu}_{1+a_{j_\rho}},$$

and

$$v_{2\rho-1} := \lambda_{1+j_{\rho-1}} - s, \quad v_{2\rho} := \lambda_{j_\rho} - s$$

for $\rho = 1, \dots, r$. Since v depends on s we also write $v(s) = v$.

Lemma 5.1: *The r -tuple u and v are highest weights with purity property, i.e. $u, v \in X_0^+(2r)$ of respective weight $\text{wt}(u) = -w$ and $\text{wt}(v) = w' + n - m - 1 - 2s$.*

Proof. Obviously u and v are dominant weights in $X^+(2r)$ since $\check{\mu}$ like μ is dominant, λ is dominant by Proposition 3.1 and we have

$$a_{j_1} < a_{j_2} < \dots < a_{j_r}$$

by the definition of jump indices. For the purity of u we have to show

$$u_{2\rho-1} + u_{2(r+1-\rho)} = -w \quad \text{for } \rho = 1, \dots, r.$$

This is an easy consequence of the purity of $\check{\mu}$, since combining the symmetry (2.4) of jump indices with the symmetry (2.3) of the position tuple a we get (remember $j_0 = 0$ and $j_r = m$)

$$(5.1) \quad a_{j_\rho} + a_{j_{1+r-\rho}} = n \quad \text{for } \rho = 1, \dots, r,$$

where we use $a_{1+j_\rho} = a_{j_{\rho+1}}$. For the purity of v we exploit the purity of $\lambda \in X_0^+(m)$ to show

$$(5.2) \quad v_{2\rho-1} + v_{2(r+1-\rho)} = w' + n - m - 1 - 2s \quad \text{for } \rho = 1, \dots, r.$$

Again the symmetry (2.4) of jump indices and Proposition 3.1 imply

$$v_{2\rho-1} + v_{2(r+1-\rho)} = \lambda_{1+j_{\rho-1}} - s + \lambda_{j_{r+1-\rho}} - s = w' + n - m - 1 - 2s,$$

as required.

Remark 5.1: *The respective defects of u and v are*

$$d(u) = \check{\mu}_{1+a_{j_{\frac{r}{2}}}} - \check{\mu}_{a_{j_{1+\frac{r}{2}}}} \quad \text{for even } r$$

and

$$d(v) = \lambda_{1+j_{\frac{r-1}{2}}} - \lambda_{j_{\frac{r+1}{2}}} \quad \text{for odd } r.$$

In particular $d(v)$ is independent of s and the existence of a critical number implies $d(u) \leq d(v)$ for even r and $d(v) \leq d(u)$ for odd r .

Proof. The explicit form of the defect can directly be read off from the definition and the inequality follows by (4.2) for the middle group.

Now with $d := \min\{d(u), d(v)\}$ we define the modified $\hat{u} := u + d \cdot l$ and $\hat{v} := v + d \cdot l$ like in the previous section. Passing to the dual of \hat{u} we find $\check{\hat{u}} = \hat{u} + w - d$.

Remark 5.2: *For the dual $\tilde{\mu} := \check{\hat{u}}$ of \hat{u} we explicitly have*

$$\tilde{\mu}_{2\rho-1} = \begin{cases} \mu_{a_{j_\rho}} - d & \text{for } 2\rho - 1 \leq r, \\ \mu_{a_{j_\rho}} & \text{for } 2\rho - 1 > r, \end{cases}$$

$$\tilde{\mu}_{2\rho} = \begin{cases} \mu_{1+a_{j_\rho}} - d & \text{for } 2\rho \leq r, \\ \mu_{1+a_{j_\rho}} & \text{for } 2\rho > r. \end{cases}$$

Proof. By Lemma 5.1 we know $wt(u) = -w$ hence \hat{u} has weight $wt(\hat{u}) = d - w$ and the dual of \hat{u} is $\tilde{\mu} = \check{\hat{u}} = \hat{u} + w - d$. By definition of \hat{u} we have

$$\hat{u}_{2\rho-1} = \begin{cases} \check{\mu}_{a_{j_\rho}} & \text{for } 2\rho - 1 \leq r, \\ \check{\mu}_{a_{j_\rho}} + d & \text{for } 2\rho - 1 > r, \end{cases}$$

and

$$\hat{u}_{2\rho} = \begin{cases} \check{\mu}_{1+a_{j_\rho}} & \text{for } 2\rho \leq r, \\ \check{\mu}_{1+a_{j_\rho}} + d & \text{for } 2\rho > r. \end{cases}$$

Since $\check{\mu} = \mu - w$ we arrive at the claimed formula for $\check{\mu}$.

In a similar way by definition of $\hat{v} = v + d \cdot l$ we have

$$\hat{v}_{2\rho-1} = \begin{cases} \lambda_{1+j_{\rho-1}} - s & \text{for } 2\rho - 1 \leq r, \\ \lambda_{1+j_{\rho-1}} + d - s & \text{for } 2\rho - 1 > r, \end{cases}$$

and

$$\hat{v}_{2\rho} = \begin{cases} \lambda_{j_\rho} - s & \text{for } 2\rho \leq r, \\ \lambda_{j_\rho} + d - s & \text{for } 2\rho > r. \end{cases}$$

Isolating the influence of s in these formulas we put $\tilde{\lambda} := \hat{v} + s$ and find the obvious explicit description of $\tilde{\lambda}$ from the preceeding lines.

Theorem A: *In the non-exceptional case the mapping $t \mapsto t + \frac{n-m}{2} - 1$ sets up a bijection*

$$\text{Crit}(\pi_\infty, \sigma_\infty) \longrightarrow \bigcap_{i=1,2} \text{Emb}(\theta'_i(\tilde{\lambda}), \theta_i(\check{\mu})).$$

Proof. For a critical pair $(\pi_\infty, \sigma_\infty)$ by Proposition 3.3 and (4.1) a number $t \in \frac{m+n}{2} + \mathbb{Z}$ is critical if and only if for $s = t + \frac{n-m}{2} - 1$ our concrete (u, v) in the beginning of this section satisfies (4.2). By Corollary 4.2 this is equivalent to the embedding property

$$\theta'_i(\hat{v}) \prec \theta_i(\hat{u}) \quad \text{for } i = 1, 2.$$

In terms of the previously defined weights $\tilde{\lambda}$ and $\check{\mu}$ this says

$$(5.3) \quad \theta'_i(\tilde{\lambda}) - s \prec \theta_i(\check{\mu}),$$

i.e.

$$s \in \bigcap_{i=1,2} \text{Emb}(\theta'_i(\tilde{\lambda}), \theta_i(\check{\mu})),$$

which settles the proof of the theorem.

Corollary 5.1: *In terms of the Tate modules $T_r(s) = \det^s$ for GL_r we have*

$$\bigcap_{i=1,2} (M_{\theta'_i(\tilde{\lambda})} \otimes M_{\theta_i(\check{\mu})})^{SL_r} \cong \bigoplus_{t \in \text{Crit}} T_r(t + \frac{n-m}{2} - 1).$$

Proof. This follows by the general principle that for $\alpha \in X_0^+(r)$ and $\beta \in X_0^+(r+1)$ we have $\alpha - s \prec \check{\beta}$ if and only if the Tate module $T_r(s)$ is a (1-dimensional) GL_r -submodule of $(M_\alpha \otimes M_\beta)^{SL_r}$.

For the remainder of this section we suppose that we are in the exceptional case. In view of Proposition 2.1 the eventual parametrization of the critical numbers will be provided by a parameter set of embeddable twists similar to the non-exceptional case subject to an additional parity condition that takes

care of $(PC)_\epsilon$. Again as before we must distinguish the two cases where the number k of jump indices of the position tuple a is even or odd (i.e. where $r = k + 1$ is odd or even).

Case $r \equiv 1 \pmod{2}$. Recall that by Proposition 3.3 the necessary condition (I) for a critical number t is equivalent to the system of inequalities (S_κ) for all $\kappa \neq \frac{r+1}{2}$ plus the middle condition as formulated in Proposition 3.3. As before an equivalent formulation is provided by a system of inequalities of the form (4.2) for highest weights $u, v \in X_0^+(2r)$ now given by

$$\begin{aligned} u_{2\rho-1} &:= \check{\mu}_{a_{j_\rho}}, \quad u_{2\rho} := \check{\mu}_{1+a_{j_\rho}} \quad \text{for } \rho = 1, \dots, \frac{r-1}{2}, \\ u_r &:= \check{\mu}_{\frac{n-1}{2}}, \quad u_{r+1} := \check{\mu}_{\frac{n+3}{2}} - 1, \\ u_{2\rho-1} &:= \check{\mu}_{a_{j_\rho}} - 1, \quad u_{2\rho} := \check{\mu}_{1+a_{j_\rho}} - 1 \quad \text{for } \rho = \frac{r+3}{2}, \dots, r, \\ v_{2\rho-1} &:= \lambda_{1+j_{\rho-1}} - s, \quad v_{2\rho} := \lambda_{j_\rho} - s \quad \text{for } \rho = 1, \dots, \frac{r-1}{2}, \\ v_r &:= \lambda_{\frac{m+1}{2}} - s =: v_{r+1}, \\ v_{2\rho-1} &:= \lambda_{1+j_{\rho-1}} - 1 - s, \quad v_{2\rho} := \lambda_{j_\rho} - 1 - s \quad \text{for } \rho = \frac{r+3}{2}, \dots, r. \end{aligned}$$

The same arguments as in the proof of Lemma 5.1 together with Proposition 3.1 show

Lemma 5.2: *The weights u and v belong to $X_0^+(2r)$ and have weight*

$$wt(u) = -w - 1, \quad wt(v) = w' + n - m - 2 - s.$$

By definition here v has defect $d(v) = 0$, i.e. $v \in X_{sp}(2r)$ and therefore by Proposition 4.1 the pair (u, v) satisfies (4.2) if and only if we have the embeddings

$$\theta'_i(v) \prec \theta_i(u) \quad \text{for } i = 1, 2.$$

Like in the non-exceptional case using the notation $\tilde{\lambda} := v + s$ (which is independent of s) and $\tilde{\mu} := \check{u}$ this again says

$$(5.3) \quad \theta'_i(\tilde{\lambda}) - s \prec \theta_i(\tilde{\mu}) \quad \text{for } i = 1, 2.$$

Case $r \equiv 0 \pmod{2}$. Here condition (I) again by Proposition 3.3 is equivalent to the system of inequalities (S_κ) for all $\kappa \neq \frac{r}{2}$ plus the truncated series of inequalities $(S'_{\frac{r}{2}})$. Thus we are led to define

$$\begin{aligned} u_{2\rho-1} &:= \check{\mu}_{a_{j_\rho}}, \quad u_{2\rho} := \check{\mu}_{1+a_{j_\rho}} \quad \text{for } \rho = 1, \dots, r, \\ v_{2\rho-1} &:= \lambda_{1+j_{\rho-1}} - s, \quad v_{2\rho} := \lambda_{j_\rho} - s \quad \text{for } \rho \neq \frac{r}{2}, \\ v_{r-1} &:= \lambda_{1+j_{\frac{r}{2}-1}} - s, \quad v_r := \lambda_{j_{\frac{r}{2}}} - s. \end{aligned}$$

Lemma 5.3: *The weights u and v belong to $X_0^+(2r)$ and moreover the defect $d(u)$ vanishes.*

Proof. The weight u is the same like in the non-exceptional case in Lemma 5.1. The weight v is almost the same up to v_r where purity follows by

$$v_r + v_{r+1} = \lambda_{j_{\frac{r}{2}}-1} - s + \lambda_{1+j_{\frac{r}{2}}} - s = \lambda_{\frac{m-1}{2}} + \lambda_{\frac{m+3}{2}} - 2s = w' + n - m - 1 - 2s,$$

and by Proposition 3.1 since by Remark 2.4 we have $j_{\frac{r}{2}} = \frac{m+1}{2}$. As to the defect we show that $1 + a_{j_{\frac{r}{2}}} = a_{j_{1+\frac{r}{2}}}$ which implies $u_r = u_{r+1}$. By Remark 2.4 the index $j = \frac{m-1}{2}$ is not a jump index for even r , so by Lemma 2.5 we have

$$a_{\frac{m-1}{2}} = a_{\frac{m+1}{2}} (= a_{j_{\frac{r}{2}}}) = \frac{n-1}{2}.$$

Since we know that $a_{j_{1+\kappa}} = a_{1+j_{\kappa}}$ for any jump index j_{κ} we get by (2.3) for $j = \frac{m-1}{2}$ the identity

$$a_{j_{1+\frac{r}{2}}} = a_{1+j_{\frac{r}{2}}} = a_{\frac{m+3}{2}} = n - a_{\frac{m-1}{2}} = \frac{n+1}{2} = 1 + a_{j_{\frac{r}{2}}}$$

as required.

Again we can apply Proposition 4.1 to $\tilde{\mu} := \check{u}$ for $u \in X_{sp}(2r)$ and $\tilde{\lambda} := v + s \in X_0^+(2r)$, hence we get that (u, v) satisfies (4.2) if and only if we have the embeddings

$$(5.3) \quad \theta'_i(\tilde{\lambda}) - s \prec \theta_i(\check{\mu}) \quad \text{for } i = 1, 2$$

just like in all other previous cases.

In order to take care of the parity condition $(PC)_{\epsilon}$ in Proposition 2.1 we introduce the notation

$$Z_{\epsilon} := (2\mathbb{N} - \epsilon) \cup -(2\mathbb{N} - 1 - \epsilon) \quad \text{for } \epsilon = 0, 1$$

and we put

$$\text{Emb}(\alpha, \beta)_{\epsilon} := \{s \in \text{Emb}(\alpha, \beta); s - \frac{1}{2}(n - m + w + w') + 1 \in Z_{\epsilon}\}$$

and

$$(M_{\alpha} \otimes M_{\beta})_{\epsilon}^{SL_r} := \bigoplus_{s \in \text{Emb}(\alpha, \check{\beta})_{\epsilon}} T_r(s).$$

The preceding discussion now eventually emerges into

Theorem B: *In the exceptional case the mapping $t \mapsto t + \frac{n-m}{2} - 1$ sets up a bijection*

$$\text{Crit}(\pi_{\infty}, \sigma_{\infty}) \longrightarrow \bigcap_{i=1,2} \text{Emb}(\theta'_i(\tilde{\lambda}), \theta_i(\check{\mu}))_{\epsilon}.$$

The analogue of the corollary of Theorem A may be formulated as follows.

Corollary 5.2: *In terms of the Tate modules $T_r(s)$ we have*

$$\bigcap_{i=1,2} (M_{\theta'_i(\tilde{\lambda})} \otimes M_{\theta_i(\tilde{\mu})})_\epsilon^{SL_r} \cong \bigoplus_{t \in \text{Crit}} T_r(t + \frac{n-m}{2} - 1).$$

We finish by comparing our results with previously studied cases in the literature.

Examples:

Case $n=2, m=1$: For a classical newform f of weight k lifting to an adelic function on $GL_2(\mathbb{A})$ with central character ω with infinity part $\omega_\infty(x) = x^k$ for $x > 0$ will provide us with an attached cohomological cuspidal automorphic representation $\pi \in \text{Coh}(\mu)$ with $\pi_\infty = J(-k, k-1) \otimes \text{sign}^\delta$ and $\mu = (k-1, 1)$. For trivial σ we have $\nu = 0, a = 1, r = 1$ and $d = 0$, hence $\tilde{\mu} = \mu$ and $\tilde{\lambda} = (0, 0)$. So Theorem A yields the bijection

$$\text{Crit} \longrightarrow \text{Emb}(0, \tilde{\mu}), \quad t \mapsto t - \frac{1}{2}$$

since $\theta'_i(\tilde{\lambda}) = \nu$ and $\theta_i(\tilde{\mu}) = \mu$ which is a general fact in the case $m = n-1$ (see below). Since for the completed L -function of the newform we have

$$L(\pi, s) = L(f, s + k - \frac{1}{2}),$$

the critical numbers are those studied by Shimura [10].

Case $n=m=2$: For two classical newforms f and g of respective weight k and l with $k > l$ we may as in the previous case attach cohomological cuspidal automorphic representations π and σ . Since $a = (1, 1)$ and $r = 1$ we get $\lambda = (l-1, 0)$ and $d = \lambda_1 - \lambda_2 = l-1$, hence we have $\tilde{\mu} = (k-l, 1), \tilde{\lambda} = (l-1, l-1), \theta'_i(\tilde{\lambda}) = l-1$ and $\theta_i(\tilde{\mu}) = \tilde{\mu} = (k-l, 1)$. We find

$$M_{\theta'_i(\tilde{\lambda})} \otimes M_{\theta_i(\tilde{\mu})} \cong M_{(k-1, l)} \cong \bigoplus_{s=l}^{k-1} T(s) \cong \bigoplus_{t \in \text{Crit}(\pi_\infty, \sigma_\infty)} T(t-1).$$

Again we encounter the critical numbers studied by Shimura [loc.cit.] for the associated L -function attached to the two forms f and g .

Case $m=n-1$: We revisit the discussion in [7] where we assumed that $\text{Emb}(\nu, \tilde{\mu})$ is non-empty. In that situation we have $a_j = j, j_\kappa = \kappa, r = m$ and $\lambda = \nu$. Moreover we get $d = 0, \theta'_i(\tilde{\lambda}) = \nu$ and $\theta_i(\tilde{\mu}) = \mu$ so by Theorem A we have the bijection

$$\text{Crit}(\pi_\infty, \sigma_\infty) \longrightarrow \text{Emb}(\nu, \tilde{\mu}), \quad t \mapsto t - \frac{1}{2}$$

and

$$(M_\mu \otimes M_\nu)^{SL_{n-1}} \cong \bigoplus_{t \in \text{Crit}} T_{n-1}(t - \frac{1}{2}).$$

Case $n = 3$, $m = 1$ The Jacquet-Gelbart-Lift: Each cuspidal automorphic representation π of GL_2 attached to a newform f of weight k as in the first example gives rise to an automorphic representation Π of GL_3 by the work of Gelbart and Jacquet [3]. The infinity component of Π is

$$\Pi_\infty \cong J(0; (2(k-1), 0, -2(k-1))) \otimes \text{sgn}$$

(see for instance [9] (1.8)). Hence for $\mu := (k-2, 0, 2-k)$ we find that Π belongs to $\text{Coh}(\mu)$ by [1] proof of Lemme 3.14. So we can apply Theorem B to the pair $(\Pi_\infty, \sigma_\infty)$ for the trivial GL_1 -representation $\sigma = \mathbf{1}$. Since $\lambda = \nu = 0$ and $d = 0$ we have $\tilde{\mu} = (k-1, 2-k)$ and $\tilde{\lambda} = (0, 0)$. So we get with $\theta'_i(\tilde{\lambda}) = 0$ and $\theta_i(\tilde{\mu}) = \tilde{\mu} = (k-1, 2-k)$

$$\text{Crit}(\Pi_\infty) = \{2-k, \dots, k-1\} \cap (\{1, 3, 5, \dots\} \cup \{0, -2, -4, \dots\})$$

as considered in Lemma 2.1 of [9]. Moreover for any größencharacter ξ with infinity component $\xi_\infty = \text{sgn} \otimes |\cdot|_{\mathbb{R}}$ we have a cohomological GL_1 -representation $\sigma \in \text{Coh}(\nu)$ for $\nu = -1$ hence with the attached $\lambda = -1$, $\tilde{\lambda} = (-1, -1)$, $\theta'_i(\tilde{\lambda}) = -1$, $\theta_i(\tilde{\mu}) = \tilde{\mu} = (k-1, 2-k)$ again by Theorem B we find

$$\text{Crit}(\Pi_\infty, \sigma_\infty) = \{1-k, \dots, k-2\} \cap (\{1, 3, 5, \dots\} \cup \{-2, -4, \dots\})$$

in accordance with Lemma 2.1 of [9] since we have

$$L(\Pi_\infty \otimes \text{sgn}, s+1) = L(\Pi_\infty, \sigma_\infty, s).$$

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